Math 214 Lecture Notes

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Contents

1 Smooth Manifolds

1.1 Topological Manifolds

Definition 1.1. Let X be a set. Furthermore, define a set τ whose elements are subsets of X such that

- $\varnothing, X \in \tau$
- $A, B \in \tau \implies A \cap B \in \tau$
- If $A_\alpha \in \tau$ for $\alpha \in I$, then $\bigcup_{\alpha \in I} A_\alpha \in \tau$.

Then, (X, τ) is a topological space. ◇

Sets in τ are referred to as *open sets*. Subsets of a topological space are *closed* if their complements are open. Furthermore, A is a neighborhood of $p \in X$ if there exists an open set containing p that is contained in A.

Definition 1.2. Let X, Y be topological spaces. Then, a function $f : X \to Y$ is *continuous* if preimages of open sets are open. ◇

Definition 1.3. A function $f : X \to Y$ is a *homeomorphism* if f is invertible and both f and its inverse are continuous. ◇

Definition 1.4. Let (X, τ_X) be a topological space and let $Y \subseteq X$. Then, Y inherits a topology from X, defined as

$$\tau_Y = \{U \cap Y \mid U \in \tau_X\}. \qquad \diamond$$

If $Y \subseteq X$, then the inclusion map from Y to X is continuous with respect to the subspace topology. Furthermore, the subspace topology is the coarsest topology (fewest open sets) for which the inclusion is continuous.

Definition 1.5. A topological space X is *Hausdorff* if any two distinct points in the space have neighborhoods that separate the two points (i.e. the neighborhoods do not intersect). ◇

Just as in metric spaces, we can define a notion of convergence on general topological spaces.

Definition 1.6. A sequence of points (x_i) converges to x if for any $U \in X$ containing x, $U \supset \{x_i\}_{i=N}^{\infty}$ for some N. ◇

Theorem 1.1. *If X is Hausdorff, then limits of convergent sequences are unique.*

Theorem 1.2. *If X is a topological space such that for any two points p, q, there exists a real-valued continuous function f such that $f(p) \neq f(q)$, then X is Hausdorff.*

Definition 1.7. A topological space X is *second-countable* if there exists a finite/countable collection of open subsets of X that generates the topology of X. ◇

Definition 1.8. A space X is *locally Eulidean of dimension n* if for every $p \in X$, there exists an open neighborhood U of p and an open $V \in \mathbb{R}^n$ such that $U \cong V$ when both are equipped with the subspace topology ◇

Observe that we can replace V in the definition above with the unit open ball in $\mathbb{R}^n$.

Definition 1.9. A topological space X is a *topological manifold of dimension n* if X is Hausdorff, second-countable, and locally Euclidean of dimension n. ◇

The condition that X is Hausdorff is necessary; for example, the line with two origins is both second-countable and locally Euclidean with dimension 1, but it is not Hausdorff.

The second-countable condition is also necessary; consider any uncountable set S with the discrete topology and define $X = S \times R$ (equivalently, the disjoint union of an uncountable number of real lines). This set X is clearly Hausdorff and locally Euclidean, but not second-countable. A connected counterexample is the long line.

Theorem 1.3. *If M^n is a topological manifold and $M' \subseteq M$ is open, then M' is an n-dimensional topological manifold.*

Definition 1.10. Let M^n be a topological manifold. A *(coordinate) chart* on M is a pair (U, φ) such that U is open in M and $\varphi : U \to \tilde{U}$ is a homeomorphism to an open subset of $\mathbb{R}$. U is referred to as a *coordinate domain.* ◇

Note that a manifold is the union of all its coordinate domains. We can then write down the *local coordinates* of a point as $\varphi(p) = (\varphi^1(p), \ldots, \varphi^n(p))$. We also refer to φ^{-1} as a *local parametrization.*

Other examples of manifolds include $\mathbb{R}^n$, S^n, and $\mathbb{RP}^n$.

Definition 1.11. A space X is *connected* if X and $\varnothing$ are the only clopen subsets of X. ◇

Definition 1.12. A space X is *path-connected* if for any $p, q \in X$, there exists a continuous function $\gamma : [0, 1] \to X$ such that $\gamma(0) = p$ and $\gamma(1) = q$. ◇

Path-connectedness implies connectednesss.

Definition 1.13. A space X is locally path-connected if every point has a path-connected, open neighborhood. ◇

Theorem 1.4. *Let M^n be a topological manifold. Then,*

- *M is locally path-connected.*
- *M is connected if and only if it is path connected.*
- *The components of M are the same as the path components.*

Theorem 1.5. *There are countably many charts (U_i, φ_i) for any topological manifold M such that*

$$\varphi_i(U_i) = B_1(0) \in \mathbb{R}^n$$

and $M = \bigcup_{i=1}^{\infty} U_i$.

Lemma 1.1. *If X is a second-countable topological space, then any open cover of X has a countable subcover.*

1.2 Smooth Structure

Definition 1.14. If $(U, \varphi), (V, \psi)$ are charts of a topological manifold M, then $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called a *transition map* (or alternatively, a *change of coordinates map*). ◇

Theorem 1.6. *Transition maps are homeomorphisms.*

Note that homeomorphisms may not preserve smoothness. Consider two charts on $\mathbb{R}^n$ (treated as a manifold), (U, id) and (V, α^{-1}), where $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism. Now, consider a function $f : \mathbb{R}^n \to \mathbb{R}$ defined on the manifold. The problem is that α may distort the space in a way such that the function $f \circ \alpha$ is no longer smooth.

Definition 1.15. Two charts are *smoothly compatible* if the transition map between them is a C^∞ diffeomorphism. ◇

Definition 1.16. An *atlas* $\mathcal{A}$ of a topological manifold M is a collection of charts of M that covers the manifold. ◇

Definition 1.17. An atlas is *smooth* if every pair of charts in the atlas is smoothly compatible. ◇

Definition 1.18. An atlas $\mathcal{A}$ is a *maximal smooth atlas* if there exists no other smooth atlas containing $\mathcal{A}$. ◇

Theorem 1.7. *Every smooth atlas of a manifold is contained in a unique maximal smooth atlas $\mathcal{A}$.*

We can replace smoothness in the theorems above by different differentiability classes (i.e. C^k, C^ω). We could even think about charts that map into $\mathbb{C}^n$, giving rise to complex manifolds.

Definition 1.19. A maximal smooth atlas $\mathcal{A}$ on a topological manifold M is called a *smooth structure*. The pair $(M, \mathcal{A})$ is referred to as a *smooth manifold*, and any chart in the atlas is referred to a *smooth chart*. ◇

Example 1.1. A trivial example of a smooth manifold is $\mathbb{R}^n$, where we take the maximal atlas to be that which contains the chart $(\mathbb{R}^n, \mathrm{id})$. △

Example 1.2. Let V be a finite-dimensional vector space, and define

$$\mathcal{A} = \left\{(V, \varphi) \;\middle|\; \varphi : V \to \mathbb{R}^{\dim V} \text{ is a linear isomorphism}\right\}.$$

Define $\overline{\mathcal{A}}$ be the maximal atlas containing $\mathcal{A}$. Then, $(V, \overline{\mathcal{A}})$ is a smooth manifold. △

We can also construct smooth manifolds from certain subsets of smooth manifolds.

Theorem 1.8. *Let $(M, \mathcal{A})$ be a smooth manifold, and let $M' \subseteq M$ be an open subset. Define $\mathcal{A}' = \{(U, \varphi) \in \mathcal{A} \mid U \subseteq M'\}$. Then, $(M', \mathcal{A}')$ is also a smooth manifold.*

Note, however, that the maximal atlas is not unique.

Example 1.3. Consider $\mathbb{R}$, and two maximal atlases

$$\mathcal{A} = \{\text{maximal atlas containing } (\mathbb{R}, \mathrm{id})\} \quad \text{and} \quad \mathcal{A}' = \left\{\text{maximal atlas containing}(\mathbb{R}, x \mapsto x^3)\right\}.$$

Observe that $x \mapsto x^3$ is not a diffeomorphism, and so the the identity map and this map are not smoothly compatible. Therefore, $\mathcal{A} \neq \mathcal{A}'$. △

We can also construct smooth manifolds by taking products of smooth manifolds.

Theorem 1.9. *Let $M_1^{n_1}, M_2^{n_2}, \ldots, M_m^{n_m}$ be smooth manifolds. Then, the product $\prod_{i=1}^m M_i^{n_i}$ is a smooth manifold.*

We now present a lemma for constructing smooth manifolds

Lemma 1.2. *Let M be a set, and $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ a family of injective maps $\varphi : U_\alpha \to \mathbb{R}^n$ for some fixed n. Furthermore, assume that*

- *For any $\alpha, \beta \in I$, $\varphi(U_\alpha \cap U_\alpha) \subseteq \mathbb{R}^n$ is open.*
- *For any α, β, the map $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is smooth.*
- *M is covered by countably many U_α.*
- *For every $p, q \in M$, $p \neq q$, either*
  - *(a) there exists $\alpha \in I$ such that $p, q \in U_\alpha$ or*
  - *(b) there exists $\alpha, \beta \in I$ such that $p \in U_\alpha$, $q \in U_\beta$, where U_α and U_β do not intersect.*

Then, there exists a unique topology on M and smooth structure such that all of the charts are contained in the maximal atlas.

The unique topology is given by

$$\tau = \{A \subseteq M \mid \varphi_\alpha(A \cap U_\alpha) \text{ is open in } \mathbb{R}^n\}.$$

Definition 1.20 (Grassmann manifolds)**.** We define the Grassmann manifold $\mathrm{Gr}_k(\mathbb{R}^n)$, where $0 \leq k \leq n$ as the set of linear subspaces of $\mathbb{R}^n$ of dimension k. Note that $\mathrm{Gr}_1(\mathbb{R}^n) = \mathbb{RP}^{n-1}$ ⋄

1.3 Manifolds with Boundary

Define $\mathbb{H}^n = \{x \geq 0 \mid x \in \mathbb{R}^n\}$. Recall that if A is any arbitrary subset of $\mathbb{R}^n$, then a function $f : A \to \mathbb{R}^m$ is smooth if f can be extended to a smooth map defined on an open set containing A.

Theorem 1.10. *A function $f : U \to \mathbb{R}^m$, where $U \subseteq \mathbb{H}^n$ is open, is smooth if f is continuous and smooth in $V = \mathbb{H}^{n\circ} \cap U$, and all of the partial derivatives of f on V can be continuously extended to all of U.*

Definition 1.21. A *topological manifold with boundary of dimension n* is a topological space that is Hausdorff and second-countable such that every point is contained in an open set that is homeomorphic to an open subset of $\mathbb{H}^n$. ◇

In the case of manifolds with boundary, charts (U, φ) can either be interior or boundary charts. A chart is a *interior chart* if $\varphi(U)$ is contained in the interior of $\mathbb{H}^n$. Otherwise, a chart is a *boundary chart* is $\varphi(U)$ intersects the boundary of $\mathbb{H}^n$.